Announcements

1) EC from Exam 1 due today

2) HW #3 is due Thursday

Observation: (complex solutions)

We observed

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

In order to get any complex number, just multiply by a nonnegative constant r!

$$re^{i\theta} = \underbrace{r\cos(\theta)}_{\text{Real part of a complex number}} + i\underbrace{r\sin\theta}_{\text{Imaginary part of a complex number}}$$

Suppose $y : \mathbb{R} \to \mathbb{C}$ is complex valued.

For each $t \in \mathbb{R}$, ← "t in $\mathbb{R}$"

we can write

$$y(t) = (\text{real part of } y(t)) + i\,(\text{imaginary part of } y(t))$$

Define $u, v : \mathbb{R} \to \mathbb{R}$,

$$\boxed{\begin{aligned} u(t) &= \text{real part of } y(t) \\ v(t) &= \text{imaginary part of } y(t) \end{aligned}}$$

Observation: If y is a solution to

$$ay'' + by' + cy = 0$$

where $a, b, c \in \mathbb{R}$ and y is complex-valued, then writing $y(t) = u(t) + iv(t)$,

$$\boxed{\begin{aligned} au'' + bu' + cu &= 0, \\ av'' + bv' + cv &= 0 \end{aligned}}$$

This is true since

$$y'(t) = u'(t) + i v'(t),$$

$$y''(t) = u''(t) + i v''(t),$$

so

$$0 = ay'' + by' + cy$$

$$= (au'' + bu' + cu) + i(av'' + bv' + cv)$$

This implies that u and v satisfy the equation since a complex number is zero precisely when its real and imaginary parts are zero.

Back to spring problems

A mass weighing 9 lb is attached to a spring hanging from the ceiling and comes to rest at its equilibrium position.

At $t=0$, an external force $\boxed{F(t) = 2\cos(2t)}$ lb is applied to the system.

If $k = 10$ lb/ft and $b = 1$ lb-sec/ft, find the equation of motion of the mass.

First, if the solution is y, we'll be solving

$$\boxed{2\cos(2t) = 9y'' + y' + 10y}$$

Since the weight is in equilibrium,

$$\boxed{y(0) = 0, \quad y'(0) = 0}.$$

<u>Step 1:</u> Solve homogeneous system

$$0 = 9y'' + y' + 10y,$$

let $y = e^{rt}$. We get

$$0 = 9r^2 + r + 10$$

$$r = \frac{-1 \pm \sqrt{1 - 4(10)(9)}}{18}$$

$$= \frac{-1 \pm \sqrt{-359}}{18}$$

$$= \frac{-1 \pm i\sqrt{359}}{18}$$

Then solutions are of the form

$$C_1 e^{\left(\frac{-1+i\sqrt{359}}{18}\right)t} + C_2 e^{\left(\frac{-1-i\sqrt{359}}{18}\right)t}$$

$$= C_1 e^{-\frac{t}{18}} e^{\frac{i\sqrt{359}t}{18}} + C_2 e^{-\frac{t}{18}} e^{-\frac{i\sqrt{359}t}{18}}$$

$$= C_1 e^{-\frac{t}{18}}\left(\cos\left(\frac{\sqrt{359}t}{18}\right) + i\sin\left(\frac{\sqrt{359}t}{18}\right)\right)$$

$$+ C_2 e^{-\frac{t}{18}}\left(\underbrace{\cos\left(-\frac{\sqrt{359}t}{18}\right) + i\sin\left(-\frac{\sqrt{359}t}{18}\right)}_{=\cos\left(\frac{\sqrt{359}t}{18}\right) - i\sin\left(\frac{\sqrt{359}t}{18}\right)}\right)$$

Rewriting, solutions are of the form

$$\underbrace{(C_1+C_2)}_{D_1}\left(e^{-\frac{t}{18}}\cos\left(\frac{\sqrt{359}\,t}{18}\right)\right)$$

$$+\underbrace{(C_1-C_2)i}_{D_2}\left(e^{-\frac{t}{18}}\sin\left(\frac{\sqrt{359}\,t}{18}\right)\right)$$

Real solutions: (linearly independent)

$$y_1(t)=e^{-\frac{t}{18}}\cos\left(\frac{\sqrt{359}\,t}{18}\right),$$

$$y_2(t)=e^{-\frac{t}{18}}\sin\left(\frac{\sqrt{359}\,t}{18}\right).$$

Step 2: We have solutions for the homogeneous system, $y_1(t)$ and $y_2(t)$. Solve the nonhomogeneous system

$$2\cos(2t) = 9y'' + y' + 10y$$

using variation of parameters:

We wish that a solution y_p to this equation is given by

$$\boxed{y_p(t) = y_1(t)u(t) + y_2(t)v(t)}$$

for some functions u and v.

We arrive at the equations

1) $0 = u'(t)y_1(t) + v'(t)y_2(t)$

and

2) $\frac{2\cos(2t)}{9} = u'(t)y_1'(t) + v'(t)y_2'(t).$

Multiply equation 1) by $-\frac{y_2'(t)}{y_2(t)}$, and to 2) to get

$$\frac{2\cos(2t)}{9} = u'(t)\left(\frac{y_1'(t)y_2(t) - y_2'(t)y_1(t)}{y_2(t)}\right),$$

So

$$u'(t) = \frac{2\cos(2t)\, y_2(t)}{9\left(y_1'(t) y_2(t) - y_2'(t) y_1(t)\right)}$$

Integrate to get $u(t)$.

$$y_1(t) = e^{-\frac{t}{18}} \cos\left(\frac{\sqrt{359}\, t}{18}\right),$$

$$y_2(t) = e^{-\frac{t}{18}} \sin\left(\frac{\sqrt{359}\, t}{18}\right)$$

$$y_2'(t) = -\frac{1}{18} e^{-\frac{t}{18}} \sin\left(\frac{\sqrt{359}\, t}{18}\right) + e^{-\frac{t}{18}} \cos\left(\frac{\sqrt{359}\, t}{18}\right)\left(\frac{\sqrt{359}}{18}\right)$$

$$y_1'(t) = -\frac{1}{18} e^{-\frac{t}{18}} \cos\left(\frac{\sqrt{359}\, t}{18}\right) - e^{-\frac{t}{18}} \sin\left(\frac{\sqrt{359}\, t}{18}\right)\left(\frac{\sqrt{359}}{18}\right)$$